GENERALIZED ASYMPTOTIC EULER'S RELATION
FOR CERTAIN FAMILIES OF POLYTOPES

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ABSTRACT. According to Euler's relation any polytope P has as many faces of even dimension as it has faces of odd dimension. As a generalization of this fact one can compare the number of faces whose dimension is congruent to i modulo m with the number of all faces of P for some positive integer m and for some $1 \leq i \leq m$. We show some classes of polytopes for which the above proportion is asymptotically equal to $1/m$.

INTRODUCTION

For any d-polytope P, we denote its f-vector by $f(P)$ whose i^{th} component $f_i(P)$ is the number of faces of dimension i in P for $i = -1, 0, \ldots, d$. We put $f_{-1}(P) = f_d(P) = 1$ for the improper faces $\emptyset$ and P itself and $f_i(P) = 0$, if $i > d$ or $i < -1$. A widely studied and important problem is the characterization of the set of all possible f-vectors of polytopes. A well-known necessary condition for an arbitrary $(d+2)$-tuple of integers to be the f-vector of some d-polytope is Euler's relation. We give this relation in an unusual form in order to emphasize the main idea of this paper. For any d-polytope P one has the equation

$$\sum_{i \equiv 0 \bmod 2} f_i(P) = \sum_{i \equiv 1 \bmod 2} f_i(P),$$

in other words the number of faces of even dimension in P is equal to the number of faces of odd dimension. In general, for any positive integer m and for any d-polytope P, we define its m-modular f-vector $f^m(P)$ to be the vector with i^{th} component $f_i^m(P)$ equal to the number of faces whose dimension is congruent to i modulo m ($i = -1, 0, \ldots, m-2$). In the special case $m = d+2$ the m-modular f-vector of P is equal to the f-vector of P and for $m = 1$ we obtain the number of all faces of P. Using this notation, Euler's relation can be given in the following simple form:

$$f_{-1}^2(P) = f_0^2(P).$$

The question necessarily arises from the above whether there exists some analogous relation for $m > 2$. Of course equality does not hold universally if $m > 2$. For example the 3-modular f-vector of the three dimensional cube is $(7, 9, 12)$, that is $f_i^3(C) \neq f_j^3(C)$ for $i, j \in \{-1, 0, 1\}$. Nevertheless the components of the 3-modular f-vector of the 6-simplex are approximately equal: $(22, 21, 21)$. Therefore it is reasonable to ivestigate the classes of polytopes $\mathcal{P}$ for which the following

statement holds. *For any positive integer m and for any $\varepsilon > 0$ there exists a positive K such that for all $P \in \mathcal{P}$ of dimension at least K*

$$||f^m(P)/f^1(P) - (1/m, \ldots, 1/m)|| < \varepsilon.$$

The aim of this paper is to present such classes of polytopes. The above statement probably holds for other natural classes of polytopes e.g. simple polytopes, cubical polytopes, possibly even for the class of all polytopes.

Pyramids, Bipyramids and Prisms

Let P_0 be an arbitrary $(d-1)$-polytope, let I be a segment not parallel to the affine hull of P_0. Let us recall that the vector-sum $P = P_0 + I$ is said to be a *prism* with base P_0. The convex hull of the union of P_0 and the segment I is called a *pyramid* with base P_0 if one endpoint of I is in P_0 and it is called a *bipyramid* with base P_0 if some interior point of I belongs to the relative interior of P_0.

Theorem 1. *Let P be any polytope. Let us define a sequence P_k of polytopes by the recursion*

$$(i) \quad P_0 = P$$

$$(ii) \quad P_{k+1} \text{ is a prism with base } P_k.$$

For any positive integer m

$$\lim_{k \to \infty} \frac{f^m(P_k)}{f^1(P_k)} = \Big(\frac{1}{m}, \ldots, \frac{1}{m}\Big).$$

Proof. We have the following relation between the f-vector of the prism P_k and the f-vector of its base P_{k-1} (see Grünbaum [3] 4.4):

$$f_0(P_k) = 2 \cdot f_0(P_{k-1}) + f_{-1}(P_{k-1}) - 1 \quad \text{and}$$
$$f_{i+1}(P_k) = 2 \cdot f_{i+1}(P_{k-1}) + f_i(P_{k-1}) \quad for \ i > 0.$$

Applying the above we obtain directly the following equations for the components of the corresponding m-modular f-vectors:

$$f_{-1}^m(P_k) - 1 = 2 \cdot (f_{-1}^m(P_{k-1}) - 1) + f_{m-2}^m(P_{k-1}),$$
$$f_0^m(P_k) = 2 \cdot f_0^m(P_{k-1}) + (f_{-1}^m(P_{k-1}) - 1),$$

(0.1)
$$\vdots$$

$$f_{m-2}^m(P_k) = 2 \cdot f_{m-2}^m(P_{k-1}) + f_{m-3}^m(P_{k-1}).$$

A nonnegative matrix M is said to be *doubly stochastic* if the sum of the entries in each row and in each column equals 1. Let Q denote the (unique) doubly stochastic matrix $(a_{i,j})$ of order m for which $a_{i,i+1} = 1$ for $1 \leq i \leq n-1$. Let e_1 denote the unit vector $(1, 0, \ldots, 0)$ and I the identity matrix. Using this notation, the equations 0.1 can be formulated concisely as

(0.2)
$$f^m(P_k) - e_1 = (f^m(P_{k-1}) - e_1)(2I + Q),$$

or based on P_0 by recursion 1 as

(0.3)
$$f^m(P_k) - e_1 = (f^m(P_0) - e_1)(2I + Q)^k.$$

For $m = 1$ the equation 0.3 yields $f^1(P_k) - e_1 = (f^1(P_0) - e_1) \cdot 3^k$, hence the kth element of the sequence of polytopes of the Theorem 1 can be written as

$$\frac{f^m(P_k)}{f^1(P_k)} = \frac{(f^m(P_0) - e_1)(2I + Q)^k + e_1}{(f^1(P_0) - 1) \cdot 3^k + 1}.$$

The vector $(f^m(P_0) - e_1)/(f^1(P_0) - 1)$ is a *stochastic* vector (its components sum to 1) and $(2I + Q)^k/3^k$ is a doubly stochastic matrix, therefore

(0.4)
$$\lim_{k \to \infty} \frac{f^m(P_0) - e_1}{f^1(P_0) - 1} \cdot \left(\frac{2I + Q}{3}\right)^k = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right).$$

This fact is an immediate consequence of the Corollary 2 of [5]. The congruence 0.4 implies the assertion of Theorem 1. $\qquad\square$

It can be proved in the same way that Theorem 1 remains valid if we replace "prism" with "pyramid" or "bipyramid" in condition (ii). As a special case of prisms and pyramids we can mention the d-cubes and d-simplices obtained by choosing a single point as the base P_0, therefore for any m the components of the m-modular f-vector of the d-cube (and d-simplex) are approximately equal if d is large enough.

STACKED POLYTOPES

Stacked polytopes have an important role associated with the Lower Bound Theorem (LBT, see Barnette [1], [2]) which provides a lower bound for the f-vectors of *simplicial polytopes* (polytopes all whose facets are simplices). Kalai [4] showed that the inequality $f_i(P) \geq \varphi_i(n, d)$ involved in the Lower Bound Theorem is sharp in the sense that $f_i(P) = \varphi_i(n, d)$ for any stacked d-polytope P and for any $0 < i < d$. A d-polytope P is said to be *stacked* provided there exists a sequence $P_0, P_1, \ldots, P_n$ $(0 \leq n)$ of polytopes such that for $i = 0, \ldots, n - 1$

(i) P_{i+1} is obtained by stacking a pyramid over a facet of P_i,

(ii) P_0 is a d-simplex and $P_n = P$.

Since P_0 is a d-simplex, the number of its vertices is $d + 1$, therefore P has $v = d + 1 + n$ vertices. As usual let $S(v, d)$ denote any stacked d-polytope with v vertices: although the combinatorial type of a stacked d-polytope is generally not determined by the numbers v and d, the f vector of $S(v, d)$ depends only on v and d (independently from the combinatorial type).

Theorem 2. *Let n be a fixed nonnegative integer. Let us define a sequence $(S_k)_{k=2}^{\infty}$ of stacked polytopes by $S_k = S(v, k)$, where $v = k + 1 + n$. Thus each S_k is a k-dimensional stacked polytope. For any positive integer m*

$$\lim_{k \to \infty} \frac{f^m(S_k)}{f^1(S_k)} = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right).$$

Proof. (similar to the proof of Theorem 1, but technically more difficult) We have the following relation for the components of the f-vector of S_k (see Kalai [4]):

(0.5)
$$f_i(S(v, d)) = \begin{cases} \binom{d}{i} v - \binom{d+1}{i+1} i & \text{for } 1 \leq i \leq d - 2 \\ (d - 1)v - (d + 1)(d - 2) & \text{for } i = d - 1. \end{cases}$$

It is quite difficult to deal with this relation in the given form. In order to integrate the two cases above we make a small modification on f-vectors. For the sake of this goal, we make use of the fact that

$$\left[\tbinom{d}{i}v - \tbinom{d+1}{i+1}i\right] - [(d-1)v - (d+1)(d-2)] = v - d + 1 = n \quad \text{for} \ \ i = d - 1.$$

Let R denote the (unique) doubly stochastic matrix $(a_{i,j})$ of order $d + 2$ for which $a_{i,j} = 1$ if $i + j = d + 3$ and we define $c = (n, n, 0, \ldots, 0)$. Let us give the modified face vector as $l = fR + c$. In other words we reverse the order of components of f and add n to the first two components. Later we shall show that this technical modification has no influence on the fact of the convergence of S_k. By applying relation 0.5 for modified face vectors $l(S_k)$, we have

$$(0.6) \qquad\qquad l_i(S_k) = \binom{k}{i}v - \binom{k+1}{i+1}i \quad \text{for all} \ i.$$

Equation 0.6 can be transformed into the following one:

$$
\begin{aligned}
(0.7) \qquad l_i(S_k) &= \tbinom{k}{i}(v - 1) + \tbinom{k}{i} - \tbinom{k}{i}i - \tbinom{k}{i+1}i \\
&= \left[\tbinom{k-1}{i-1}(v-1) + \tbinom{k}{i} - \tbinom{k}{i}i\right] + \left[\tbinom{k-1}{i}(v-1) - \tbinom{k}{i+1}i\right] \\
&= l_{i-1}(S_{k-1}) + l_i(S_{k-1}).
\end{aligned}
$$

We denote the modified m-*modular face vector* of S_k by $l^m(S_k)$ whose i^{th} component $l_i^m(S_k)$ is the sum of those components of $l(S_k)$ whose index is congruent to i modulo m for $i = -1, 0, \ldots, m - 2$. We obtain from 0.7 that $l^m(S_k) = l^m(S_{k-1})(I + Q)$ or recursively $l^m(S_k) = l^m(S_2)(I + Q)^{k-2}$ (compare with 0.1, 0.2 and 0.3). Therefore we find (see 0.4)

$$(0.8) \qquad\qquad \lim_{k \to \infty} \frac{l^m(S_2)}{l^1(S_2)} \cdot \left(\frac{I + Q}{2}\right)^{k-2} = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right).$$

Since $\lim(n/2^k) = 0$ as k tends to infinity and the vector $l^m(S_k) - c$ is a permutation of the vector $f^m(S_k)$, the assertion of Theorem 2 is proved. $\qquad\square$

CYCLIC POLYTOPES

The Upper Bound Theorem (proved by McMullen [6]) provides upper bound for the f-vectors of convex polytopes. This upper bound is attained by the cyclic polytopes. The cyclic polytope $C(v, d)$ is the convex hull of any v points on the moment curve $(t, t^2, \ldots, t^d) : t \in \mathbb{R}$ in $\mathbb{R}^d$. The combinatorial type of $C(v, d)$ is uniquely determined by v and d, therefore its f-vector depends only on v and d.

Theorem 3. *Let n be a nonnegative integer. Let us define a sequence $(C_k)_{k=2}^{\infty}$ of cyclic polytopes by $C_k = C(v, k)$, where $v = k + 1 + n$. For any positive integer m*

$$\lim_{k \to \infty} \frac{f^m(C_k)}{f^1(C_k)} = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right).$$

Proof. We use the notation and methods of the proof of Theorem 1 and Theorem 2 with some modification. In addition we put $A = I + Q$ (see 0.2) and agree that J denotes the $m \times m$ doubly stochastic matrix with all entries equal to $1/m$. We

have the following relation for the components of the f-vector of C_k (see McMullen [6]):

$$(0.9) \qquad f_{i-1}(C_k) = \sum_{j=0}^{i} \binom{k-j}{k-i} h_j(C_k),$$

where

$$(0.10) \qquad h_j(C_k) = \binom{v-k-1+j}{j}, \quad \text{for} \quad 0 \le j \le \left\lfloor \frac{k}{2} \right\rfloor$$

and for $\left\lfloor \frac{k}{2} \right\rfloor < j \le k$ we have $h_j(C_k) = h_{k-j}(C_k)$ from the Dehn-Sommerville equations. For the sake of manageability, we reverse the face vectors as in the proof of Theorem 2:

$$(0.11) \qquad l_{k-i}(C_k) = f_{i-1}(C_k) = \sum_{j=0}^{i} \binom{k-j}{k-i} h_j(C_k), \quad \text{for} \quad 1 \le i \le k$$

and $l_{-1}(C_k) = h_k = 1$, $l_k(C_k) = h_0 = 1$. From 0.11 one has the following recursive formula (so-called Pascal's rule):

$$(0.12) \qquad l_{k-i} = \sum_{j=0}^{i} \binom{k-j-1}{k-i-1} h_j + \sum_{j=0}^{i} \binom{k-j-1}{k-i} h_j, \quad \text{for} \quad 1 \le i \le k.$$

Now we fix the integer m. From the above we obtain the following for the modified m-modular face vector of C_k (see 0.8):

$$l^m(C_k) = ((h_0 e_1 A + (h_1 - h_0)e_1)A + \cdots + (h_k - h_{k-1})e_1)A,$$

by the convention that $h_{-1} = 0$ and using that $A^{j+1} = A^j Q + A^j$ it can be written equivalently as

$$(0.13) \qquad l^m(C_k) = e_1 \sum_{j=0}^{k} (h_j - h_{j-1})A^{k-j+1} = e_1(I + Q \cdot \sum_{j=0}^{k} h_j A^j).$$

Every matrix in this paper has the common property that the sum of the entries in each row and in each column are equal. For a matrix M with the mentioned property let $\sigma(M)$ denote this sum. Clearly $M/\sigma(M)$ is a doubly stochastic matrix. We have to prove that $\sum h_j A^j / \sigma(\sum h_j A^j)$ converges to J as k tends to infinity. It is equivalent to show that the difference between the maximum entry and minimum entry of matrix $\sum h_j A^j$ reduced by $\sigma(\sum h_j A^j)$ is arbitrarily small as k tends to infinity. Let us denote this *reduced spread* of a matrix M by $\delta(M)$. Using the notation $B := A^{2m}$ let us give $\sum h_j A^j$ in the following form:

$$A^0 \cdot (b_{0,0} I + b_{0,1} B + b_{0,2} B^2 + \cdots) +$$
$$A^1 \cdot (b_{1,0} I + b_{1,1} B + b_{1,2} B^2 + \cdots) +$$
$$\vdots$$
$$A^{2m-1} \cdot (b_{2m-1,0} I + b_{2m-1,1} B + b_{2m-1,2} B^2 + \cdots),$$

where $b_{i,j} = h_{2mj+i}$ (we agree that $h_j = 0$ if $j > k$). It is enough to show that for $i = 0, \ldots, 2m-1$ the reduced spread $\delta(c_{i,0} I + c_{i,1} B + \cdots + c_{i,s} B^s)$ (where $c_{i,s} B^s$ is the last nonzero term in the above sum) converges to zero as k tends to infinity.

We take advantage of the facts that the maximum and minimum entries are in the same place of the matrix B^j for all j and $\delta(B^i) \geq \delta(B^j)$ if $i < j$. The coefficients $c_{i,0}, c_{i,1}, \cdots, c_{i,s}$ form a unimodal sequence, that is, $c_{i,0} \leq c_{i,1} \leq \cdots \leq c_{i,r} \geq \cdots \geq c_{i,s}$ for some $r \in \{0, \ldots, s\}$. We handle the sums $c_{i,0}I + c_{i,1}B + \cdots + c_{i,r}B^r$ and $c_{i,r+1}B^{r+1} + \cdots + c_{i,s}B^s$ separately.

$$
\begin{aligned}
\delta(c_{i,0}I + c_{i,1}B + \cdots + c_{i,r}B^r) &\leq \delta(c_{i,r}I + c_{i,r}B + \cdots + c_{i,r}B^r) \leq \\
\delta(I + B + \cdots + B^r) &\leq \delta((I+B)^r)
\end{aligned}
$$
(0.14)

In order to show the following it is enough to consider that for positive a, b, c, d if $\frac{a}{c} \leq \frac{b}{d}$, then $\frac{a}{c} \leq \frac{a+b}{c+d} \leq \frac{b}{d}$:

$$
\delta(c_{i,r+1}B^{r+1} + \cdots + c_{i,s}B^s) \leq \delta(c_{i,r}B^r) \leq \delta(B^r)
$$
(0.15)

From inequalities 0.14 and 0.15 follows directly that $\delta(c_{i,0}I + c_{i,1}B + \cdots + c_{i,s}B^s) \to 0$ as k tends to infinity. $\qquad\square$

REMARKS

The following statement is an equivalent formulation of Theorem 2. *For any positive integers m, n and for any $\varepsilon > 0$ there exists a positive K such that for all stacked polytopes S of dimension $d \geq K$ with $v = d + 1 + n$ vertices*

$$
||f^m(S)/f^1(S() - (1/m, \ldots, 1/m)|| < \varepsilon,
$$

As we suggested at the end of the Introduction, it is possible that the restriction to the stacked polytopes and the stipulation $v = d + 1 + n$ may be removed.

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